SECTION 8.4: TRIGONOMETRIC SUBSTITUTIONS

SCALING THE PYTHAGOREAN IDENTITIES: Let $a > 0$ be a constant. Then:

- $a^2 - a^2\sin^2(\theta) = a^2\cos^2(\theta)$

- $a^2\tan^2(\theta) + a^2 = a^2\sec^2(\theta)$

- $a^2\sec^2(\theta) - a^2 = a^2\tan^2(\theta)$

These identities allow us to rewrite a sum or difference of squares as a single square. This is especially useful when integrands feature square roots of sums or differences of squares.

STRATEGIES: Let $a > 0$ be a constant.

- If the integrand contains $\sqrt{a^2 - x^2}$, try letting $x = a\sin(\theta)$ for $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$.

  Then $dx = a\cos(\theta)\,d\theta$ and: $\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2(\theta)} = \sqrt{a^2\cos^2(\theta)} = |a\cos(\theta)| = a\cos(\theta)$.

  Note we can drop the absolute values, $|a\cos(\theta)| = a\cos(\theta)$ since $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$.

- If the integrand contains $\sqrt{x^2 + a^2}$, try letting $x = a\tan(\theta)$ for $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$.

  Then $dx = a\sec^2(\theta)\,d\theta$ and: $\sqrt{x^2 + a^2} = \sqrt{a^2\tan^2(\theta) + a^2} = \sqrt{a^2\sec^2(\theta)} = |a\sec(\theta)| = a\sec(\theta)$.

  Note we can drop the absolute values, $|a\sec(\theta)| = a\sec(\theta)$ since $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$.

- If the integrand contains $\sqrt{x^2 - a^2}$, try letting $x = a\sec(\theta)$ for $0 \le \theta < \dfrac{\pi}{2}$ or $\dfrac{\pi}{2} < \theta \le \pi$

  Then $dx = a\sec(\theta)\tan(\theta)\,d\theta$ and: $\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2(\theta) - a^2} = \sqrt{a^2\tan^2(\theta)} = |a\tan(\theta)|$.

  If $x \ge a$ then $0 \le \theta < \dfrac{\pi}{2}$ lives in Quadrant I, so we can drop the absolute values, $|a\tan(\theta)| = a\tan(\theta)$.

  If $x \le -a$ then $\dfrac{\pi}{2} < \theta \le \pi$ lives in Quadrant II, so in this case, $|a\tan(\theta)| = -a\tan(\theta)$.

RECALL: If $a > 0$, we have the formulas below (without resorting to Trigonometric Substitution):

- $\displaystyle\int \frac{1}{\sqrt{a^2 - u^2}}\,du = \arcsin\left(\frac{u}{a}\right) + C = \sin^{-1}\left(\frac{u}{a}\right) + C$

- $\displaystyle\int \frac{1}{u^2 + a^2}\,du = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$

- $\displaystyle\int \frac{1}{u\sqrt{u^2 - a^2}}\,du = \frac{1}{a}\text{arcsec}\left(\frac{|u|}{a}\right) + C = \frac{1}{a}\sec^{-1}\left(\frac{|u|}{a}\right) + C$

EXAMPLE 1: Find the following integrals:

1. $\displaystyle\int_0^1 \sqrt{4-x^2}\,dx$ Ans: $\dfrac{\pi}{3}+\dfrac{\sqrt{3}}{2}$

2. $\displaystyle\int (x^2+25)^{-\frac{1}{2}}\,dx$ Ans: $\ln\left|\dfrac{x}{5}+\dfrac{\sqrt{x^2+25}}{5}\right|+C=\ln\left|x+\sqrt{x^2+25}\right|+C$

3. $\displaystyle\int_{\frac{3}{2}}^{\frac{3\sqrt{2}}{2}} \sqrt{4x^2-9}\,dx$ Ans: $\dfrac{9\sqrt{2}}{4}-\dfrac{9}{4}\ln\left(\sqrt{2}+1\right)$

4. $\displaystyle\int_2^4 (4x-x^2)^{\frac{1}{2}}\,dx$ Ans: π (You can check this one easily geometrically!)

EXAMPLE 2: (VIDEO) Find the following integrals:

1. $\displaystyle\int \frac{\sqrt{4-x^2}}{x}\,dx$ Ans: $\sqrt{4-x^2}+2\ln\left|\dfrac{2-\sqrt{4-x^2}}{x}\right|+C$

2. $\displaystyle\int_{-3\sqrt{2}}^{-3} \sqrt{x^2-9}\,dx$ Ans: $\dfrac{9\sqrt{2}}{2}-\dfrac{9}{2}\ln\left(\sqrt{2}+1\right)$

EXAMPLE 3: (VIDEO) Find the following integral: $\displaystyle\int (x-3)^{-\frac{3}{2}}(x+7)^{-\frac{3}{2}}\,dx$. Be mindful of quadrants!

Ans: $-\dfrac{1}{25}\,\dfrac{x+2}{\sqrt{x^2+4x-21}}+C$

HOMEWORK: Section 8.4: 9 - 55 odd, 61 - 69 odd, 81*

MATH 2500: INVERSE CIRCULAR FUNCTIONS REVIEW

- $\sin^{-1}(x) = \arcsin(x)$ is an angle between $-\pi/2$ and $\pi/2$ (Quadrant I or $-$IV) whose sine is x.
- $\csc^{-1}(x) = \operatorname{arccsc}(x)$ is an angle between $-\pi/2$ and $\pi/2$ (Quadrant I or $-$IV) whose cosecant is x.
- $\tan^{-1}(x) = \arctan(x)$ is an angle between $-\pi/2$ and $\pi/2$ (Quadrant I or $-$IV) whose tangent is x.
- $\cos^{-1}(x) = \arccos(x)$ is an angle between 0 and π (Quadrant I or II) whose cosine is x.
- $\sec^{-1}(x) = \operatorname{arcsec}(x)$ is an angle between 0 and π (Quadrant I or II) whose secant is x.
- $\cot^{-1}(x) = \operatorname{arccot}(x)$ is an angle between 0 and π (Quadrant I or II) whose cotangent is x.

1. **'Walking':** Find the **exact** values of the expressions below without using a calculator.
   (a) $\sin^{-1}\left(\frac{1}{2}\right)$
   (b) $\cot^{-1}(-1)$
   (c) $\arccos(0)$
   (d) $\tan^{-1}(\sqrt{3})$
   (e) $\sec^{-1}(-2)$
   (f) $\operatorname{arccsc}\left(-\frac{2\sqrt{3}}{3}\right)$
   (g) $\sin(\arcsin(-1))$
   (h) $\cos(\cos^{-1}(.2))$
   (i) $\arctan(\tan(5\pi/4))$
   (j) $\csc^{-1}\left(\csc\left(\frac{\pi}{3}\right)\right)$
   (k) (TRICK!) $\operatorname{arcsec}(\sec(\pi/2))$ Explain why this is a trick question.

2. **'Jogging':** Find the **exact** values of the expressions below without using a calculator.
   (a) $\sec(\arctan(-1))$
   (b) $\tan(\arccos(-1/2))$
   (c) $\cos(\cot^{-1}(-3/4))$
   (d) $\csc(\arcsin(.3))$
   (e) $\cot(\sec^{-1}(-3))$
   (f) Find an algebraic expression for $\tan(\arcsin(2x))$.

3. **'Running':**
   (a) Explain why $\csc^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right)$ and $\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$
   (b) Find a way to graph $y = \csc^{-1}(x)$ and $y = \sec^{-1}(x)$ on your calculator. (HINT: Use the result from the previous problem!)
   (c) Find $\cot^{-1}(-5)$ to four decimal places.

4. **Tournament of Champions!**
   (a) Find an algebraic expression for $\tan(\operatorname{arcsec}(x))$.
   (b) Come up with a way to use your calculator to graph $y = \cot^{-1}(x)$.

Answers.

1. (a) $\pi/6$
   (b) $3\pi/4$
   (c) $\pi/2$
   (d) $\pi/3$
   (e) $2\pi/3$
   (f) $-\pi/3$
   (g) -1
   (h) .2
   (i) $\pi/4$
   (j) $\pi/3$
   (k) $\sec(\pi/2)$ is undefined!
2. (a) $\sqrt{2}$
   (b) $-\sqrt{3}$
   (c) $-3/5$
   (d) $10/3$
   (e) $-\sqrt{2}/4$
   (f) $\dfrac{2x}{\sqrt{1-4x^2}}$
3. (a) Draw a picture or use the definition of the functions in the appropriate quadrants.
   (b) Graph $y = \sin^{-1}(1/x)$ instead of $y = \csc^{-1}(x)$ and graph $y = \cos^{-1}(1/x)$ instead of $y = \sec^{-1}(x)$.
   (c) $\cot^{-1}(-5) = \pi - \tan^{-1}(1/5) \approx 2.9442$. (Can you see why using the Unit Circle?)
4. (a) $\tan(\text{arcsec}(x)) = \sqrt{x^2-1}$, if $x > 0$ and $\tan(\text{arcsec}(x)) = -\sqrt{x^2-1}$ if $x < 0$.
   (b) You may want to break it up into pieces...